

Problem 2.29

[Difficulty: 4]

2.29 Streaklines are traced out by neutrally buoyant marker fluid injected into a flow field from a fixed point in space. A particle of the marker fluid that is at point (x, y) at time t must have passed through the injection point (x_0, y_0) at some earlier instant $t = \tau$. The time history of a marker particle may be found by solving the pathline equations for the initial conditions that $x = x_0, y = y_0$ when $t = \tau$. The present locations of particles on the streakline are obtained by setting τ equal to values in the range $0 \leq \tau \leq t$. Consider the flow field $\vec{V} = ax(1 + bt)\hat{i} + cy\hat{j}$, where $a = c = 1 \text{ s}^{-1}$ and $b = 0.2 \text{ s}^{-1}$. Coordinates are measured in meters. Plot the streakline that passes through the initial point $(x_0, y_0) = (1, 1)$, during the interval from $t = 0$ to $t = 3 \text{ s}$. Compare with the streamline plotted through the same point at the instants $t = 0, 1$, and 2 s .

Given: Velocity field

Find: Plot of streakline for $t = 0$ to 3 s at point $(1, 1)$; compare to streamlines through same point at the instants $t = 0, 1$ and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t , where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = a \cdot x \cdot (1 + b \cdot t) \quad a = 1 \quad \frac{1}{s} \quad b = \frac{1}{5} \quad \frac{1}{s} \quad v_p = \frac{dy}{dt} = c \cdot y \quad c = 1 \quad \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a \cdot (1 + b \cdot t) \cdot dt \quad \frac{dy}{y} = c \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2}\right) \quad \ln\left(\frac{y}{y_0}\right) = c \cdot (t - t_0)$

$$x = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2}\right)} \quad y = y_0 \cdot e^{c \cdot (t - t_0)}$$

The pathlines are

$$x_p(t) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2} \right)} \quad y_p(t) = y_0 \cdot e^{c \cdot (t - t_0)}$$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then

$$x_{st}(t_0) = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2} \right)} \quad y_{st}(t_0) = y_0 \cdot e^{c \cdot (t - t_0)}$$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{c \cdot y}{a \cdot x \cdot (1 + b \cdot t)}$$

So, separating variables

$$(1 + b \cdot t) \cdot \frac{dy}{y} = \frac{c}{a} \cdot \frac{dx}{x} \quad \text{which we can integrate for any given } t \text{ (} t \text{ is treated as a constant)}$$

Integrating

$$(1 + b \cdot t) \cdot \ln(y) = \frac{c}{a} \cdot \ln(x) + \text{const}$$

The solution is

$$y^{1+b \cdot t} = \text{const} \cdot x^{\frac{c}{a}}$$

For particles at (1,1) at $t = 0, 1$, and 2 s

$$y = x \quad y = x^{\frac{2}{3}} \quad y = x^{\frac{1}{2}}$$

